

COMPLEX VECTOR LATTICES VIA FUNCTIONAL COMPLETIONS

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ABSTRACT. We show that the Fremlin tensor product $C(X) \bar{\otimes} C(Y)$ is not square mean complete when X and Y are uncountable metrizable compact spaces. This motivates the definition of complexification of Archimedean vector lattices, the Fremlin tensor product of Archimedean complex vector lattices, and a theory of powers of Archimedean complex vector lattices.

1. Introduction

The standard references for the theory of vector lattices and Banach lattices (see [14], [16], [21], and [26]) all devote some attention to complex vector lattices and complex Banach lattices, but a reading of the treatment makes one feel that something is amiss. The existence of a real cone in a complex vector space did show early promise, and in fact, is essential at times in topics ranging from spectral theory and vector measures to harmonic analysis. The emerging idea of a complex modulus in the vector space complexification $E + iE$ of a Banach lattice E dates to a 1963 paper by Rieffel (see [19] and also [20]) dealing with complex AL-spaces. In 1968 (see [13]), Lotz defined, more generally, for Banach lattices E and all $f, g \in E$ the modulus $|f + ig|$ of an element $f + ig \in E + iE$ by

$$|f + ig| = \sup\{f \cos \theta + g \sin \theta : 0 \leq \theta \leq 2\pi\}. (*)$$

Luxemburg and Zaanen extend formula (*) above to all uniformly complete vector lattices (in [15]) in 1971, while studying order bounded maps and integral operators. They realized that a theory of vector lattices over \mathbb{C} has to include a complex version of the Kantorovich formula for the modulus of operators in the space of order bounded operators $E \rightarrow F$, denoted by $\mathcal{L}_b(E, F)$, when E is an Archimedean vector lattice and F is Dedekind complete.

Date: October 23, 2014.

2010 Mathematics Subject Classification. 46A40, 46B40, 46G25.

Key words and phrases. vector lattices, complexifications, Fremlin tensor product, multilinear maps, order bounded variation.

That very building block was provided in 1973 by de Schipper in [24], with the existence of the supremum in $(*)$ as a condition on E and Dedekind completeness of F as follows. By defining a space of complex order bounded operators $\mathcal{L}_b(E + iE, F + iF)$, de Schipper proved that

$$\mathcal{L}_b(E, F) + i\mathcal{L}_b(E, F) = \mathcal{L}_b(E + iE, F + iF).$$

Using the subscript \mathbb{C} for the complexification of a vector space, he thus proved that

$$\mathcal{L}_b(E_{\mathbb{C}}, F_{\mathbb{C}}) = \mathcal{L}_b(E, F)_{\mathbb{C}}.$$

Interestingly, Luxemburg and Zaanen had proved the complex Kantorovich formula in the earlier paper [15], mentioned above, under the stronger condition that E is uniformly complete. Schaefer, in his book ([21]), defines complex vector lattices axiomatically and derives formula $(*)$, but includes uniform completeness in the axioms as well.

In spite of the validity of de Schipper's theorem under the mere assumption of $(*)$, the assumption of uniform completeness has proliferated in studies on complex vector lattices, almost invariably identified with complexifications $E + iE$ of uniformly complete vector lattices E . The choice of definition for complex vector lattices in [21] as well as the standard assumption of uniform completeness in results for complex vector lattices in [26] appears to have codified that practice.

However, an alternative does exist in the literature, though it has hardly been used. Indeed, Mittelmeyer and Wolff in 1974 (see [17]) define what we call Archimedean complex vector lattices by axiomatizing an Archimedean modulus and they show that the resulting Archimedean complex vector lattices are exactly the ones that are vector space complexifications of Archimedean vector lattices with property $(*)$. In light of the history sketched above, their complex Archimedean vector lattices provide a ready made utility. The reader might well ask: Why then write this paper?

One answer simply is this. Rewriting all of the theory for results that are valid in Archimedean real vector lattices and Archimedean complex vector lattices alike, seems a rather Herculean and, at times, uninteresting task. We hasten to add that fundamental results for real vector lattices exist that are not valid for complex ones. An example is the Riesz decomposition property (see [25]). In the opposite direction, Kalton recently (see [11]) proved surprising results for complex Banach lattices that fail for real Banach lattices. In between there is a large body of results that both theories have in common. But, even with complex vector lattices satisfactorily defined in [17], these results that are in common, lack a proper transfer mechanism, a more or less mechanical procedure that transfers real results into their complex analogues, like de Schipper's result above.

In this paper, we present exactly such a mechanism. We do this in three ways. First, we construct a vector lattice complexification for every Archimedean real vector lattice, moving away from the vector space complexification for which one needs to know a priori that one deals with a vector lattice in which formula (*) is valid. Secondly, these new complexifications are precisely the Archimedean complex vector lattices introduced by Mittelmeyer and Wolff. Thirdly, we show that these newly constructed complexifications satisfy a natural universal property which, in many instances, tremendously facilitates the transfer mechanism from real results to complex results. We introduce this vector lattice complexification with a purpose in mind: differentiation in Archimedean complex vector lattices via multilinear maps and tensor products. The real version of such differentiation in vector lattices was introduced by Loane in [12]. A rapid development of polynomials on vector lattices is currently under way and complex tensor products and complex powers of complex vector lattices are needed. Motivated initially by this attempt to complex differentiation, we started by looking at the real Fremlin tensor product $E\bar{\otimes}E$ and were willing to assume uniform completeness of E , which has been the *modus operandi* in the literature, in order for

$$E\bar{\otimes}E + i(E\bar{\otimes}E)$$

to be a complex vector lattice. However, $C(X)\bar{\otimes}C(Y)$ for any uncountable compact metrizable spaces X and Y fails to have property (*) above. Introspection of the meaning of this failure naturally leads to the concept of a new vector lattice complexification to address this deficiency.

This brings us to the content of this paper.

We indeed adopt the notion of a modulus on a vector space over \mathbb{R} or \mathbb{C} , as introduced by Mittelmeyer and Wolff, in order to have uniformity in language for results that are valid for both real and complex vector lattices.

We have focused in this paper on results that are valid for complex and real vector lattices. Indeed, we use the square mean completion and its close ally, the vector lattice complexification to which we alluded above, to construct a variety of Archimedean vector lattices over \mathbb{R} or \mathbb{C} , including the Fremlin tensor product of Archimedean complex vector lattices, powers of Archimedean complex vector lattices, and Archimedean complex vector lattices of maps of order bounded variation. These spaces in turn generalize a host of results known for vector lattices over \mathbb{R} to vector lattices over \mathbb{R} or \mathbb{C} , including a generalization of de Schipper's result to multilinear maps of order bounded variation.

Finally, from our introduction above, the reader can rightly infer that a literature search will find instances where uniform completeness was used by habit rather than necessity to employ the vector space complexification in results on complex vector lattices. To find such instances has not been a purpose of the present paper. Instead, the methods in this paper provide an avenue to begin a more systematic transfer from the vast literature on Archimedean real vector lattices to the, in comparison, meagre set of results for Archimedean complex vector lattices.

The first author gladly acknowledges a conversation with Arnoud van Rooij, some twenty years ago, in which the idea and potential for a vector lattice complexification were first raised.

2. Preliminaries

For all unexplained terminology about vector lattices we refer the reader to the standard texts [1], [14], and [26]. Throughout, \mathbb{K} stands for either \mathbb{R} or \mathbb{C} , whereas \mathbb{N} stands for the (nonzero) positive integers. For $s \in \mathbb{N}$, we write $\times_{k=1}^s A_k$ for the Cartesian product $A_1 \times \cdots \times A_s$, while $A \times \cdots \times A$ (s times) is denoted by $\times_s A$.

For the definition of an Archimedean vector lattice over \mathbb{K} , central to this paper, we need the notion of a modulus on a vector space (Mittelmeyer and Wolff in [17]).

Definition 2.1. *A modulus on a vector space E over \mathbb{K} is an idempotent mapping m on E that satisfies*

- (1) $m(\alpha f) = |\alpha| m(f)$ for every $\alpha \in \mathbb{K}$ and for every $f \in E$,
- (2) $m(m(m(f) + m(g)) - m(f + g)) = m(f) + m(g) - m(f + g)$ for every $f, g \in E$, and
- (3) E is in the \mathbb{K} -linear hull of $m(E)$.

A modulus m is said to be Archimedean if for $f, g \in E$ it follows from $m(m(g) - nm(f)) = m(g) - nm(f)$ for every $n \in \mathbb{N}$ that $f = 0$.

We summarize the facts, obtained by Mittelmeyer and Wolff, in the following theorem.

Theorem 2.2. ([17], Lemma 1.2, Corollary 1.4, Proposition 1.5, Theorem 2.2)

- (1) *If m is a modulus on a vector space E over \mathbb{R} then $m(E)$ is a cone in E . Moreover, E is a vector lattice (as defined in [1] and [14]) under the partial ordering induced by $m(E)$. Furthermore, $m(f) = f \vee (-f)$ for every $f \in E$.*
- (2) *If m is an Archimedean modulus on a vector space E over \mathbb{C} then E is of the form $E_\rho \oplus iE_\rho$, where $E_\rho := m(E) - m(E)$ is an Archimedean vector lattice under the*

partial ordering induced by $m(E)$. Moreover, $\sup\{f \cos \theta + g \sin \theta : \theta \in [0, 2\pi]\}$ exists in E_ρ for every $f, g \in E_\rho$. Also, $m(f + ig) = \sup\{f \cos \theta + g \sin \theta : \theta \in [0, 2\pi]\}$ for every $f + ig \in E$.

We call a vector space over \mathbb{K} that is equipped with an Archimedean modulus m an *Archimedean vector lattice over \mathbb{K}* . Since at times we will prove results for Archimedean vector lattices over \mathbb{K} that are known for Archimedean vector lattices over \mathbb{R} , we will use absolute value signs rather than m , for convenience. The latter is justified by Theorem 2.2 above and, in turn, it enables us from here on to talk about an Archimedean vector lattice over \mathbb{K} while suppressing the modulus m .

Let E be an Archimedean vector lattice over \mathbb{K} . We define $E^+ := \{f \in E : |f| = f\}$ and call E^+ the *positive cone of E* . Then $E_\rho = E^+ - E^+$ by Theorem 2.2. A subset A of an Archimedean vector lattice E over \mathbb{K} is said to be *order bounded* if there exists $f \in E^+$ such that $|a| \leq f$ for every $a \in A$. If $\sup A$ exists in E^+ for every nonempty order bounded subset A of E^+ , we call E *Dedekind complete*. If E is an Archimedean vector lattice over \mathbb{R} , we denote by E^δ the Dedekind completion of E .

Given Archimedean vector lattices E and F over \mathbb{K} , a \mathbb{K} -linear map $T : E \rightarrow F$ is called a *vector lattice homomorphism* if $T(|f|) = |T(f)|$ for every $f \in E$. A bijective vector lattice homomorphism is called a *vector lattice isomorphism*. If there exists a vector lattice isomorphism between E and F we say that E and F are *isomorphic as vector lattices over \mathbb{K}* .

More generally, let V be a vector space over \mathbb{R} . We call V the *real part* of the complex vector space $V \oplus iV$. In the latter case, we write $V = (V \oplus iV)_\rho$, in accordance with the notation in Mittelmeyer and Wolff's Theorem 2.2 above. The complex vector space $V_\mathbb{C} := V \oplus iV$ is called the *vector space complexification* of V . As usual, we consider V to be a subset of $V_\mathbb{C}$ via the natural embedding. Given vector spaces V_1, \dots, V_s, W over \mathbb{R} and a map $T : \times_{k=1}^s V_{k\mathbb{C}} \rightarrow W_\mathbb{C}$, we say that T is *real* if $T(f_1, \dots, f_s) \in W$ whenever $f_k \in V_k$ ($k \in \{1, \dots, s\}$). The following formula (see Theorem 3 in [4]) uniquely extends a map $T : \times_{k=1}^s V_k \rightarrow W$ which is linear over \mathbb{R} in each variable separately (for short, *s \mathbb{R} -linear*) to a map $T_\mathbb{C} : \times_{k=1}^s V_{k\mathbb{C}} \rightarrow W_\mathbb{C}$ which is linear over \mathbb{C} in each variable separately (for short, *s \mathbb{C} -linear*):

$$T_\mathbb{C}(f_0^1 + if_1^1, \dots, f_0^s + if_1^s) = \sum_{\epsilon_k \in \{0,1\}} T(f_{\epsilon_1}^1, \dots, f_{\epsilon_s}^s) i^{\sum_{k=1}^s \epsilon_k}$$

where $(f_0^1 + if_1^1, \dots, f_0^s + if_1^s) \in \times_{k=1}^s V_{k\mathbb{C}}$. We will say that $T_\mathbb{C}$ is the *complexification of T* . Conversely, when $T : \times_{k=1}^s V_{k\mathbb{C}} \rightarrow W_\mathbb{C}$ is a, not necessarily separately linear, real map then

we write T_ρ for the restriction of T to $\times_{k=1}^s V_k$. It follows that $(T_\rho)_\mathbb{C} = T$ whenever T is a real $s_\mathbb{C}$ -linear map. We point out (see [22]) that every vector space over \mathbb{C} can be written as $V \oplus iV$ for some vector space V over \mathbb{R} . An Archimedean vector lattice over \mathbb{C} , however, contains a canonical real part that is determined by its modulus. This fact has a variety of consequences because much of the basic theory of Archimedean vector lattices over \mathbb{R} is encoded via vector lattice homomorphisms and positive linear maps and runs parallel with the theory of Archimedean vector lattices over \mathbb{C} . We collect some examples of this phenomenon that will be used repeatedly.

Let E_1, \dots, E_s, F be Archimedean vector lattices over \mathbb{K} . A map $T : \times_{k=1}^s E_k \rightarrow F$ is called *positive* if $T(f_1, \dots, f_s) \in F^+$ whenever $f_k \in E_k^+$ for all $k \in \{1, \dots, s\}$. An $s_\mathbb{K}$ -linear map $T : \times_{k=1}^s E_k \rightarrow F$ is a map which is linear over \mathbb{K} in each variable separately. An $s_\mathbb{K}$ -linear map $T : \times_{k=1}^s E_k \rightarrow F$ is called a *vector lattice s -morphism* if for each $k \in \{1, \dots, s\}$ the map from E_k to F defined by $f_k \mapsto T(f_1, \dots, f_k, \dots, f_s)$ is a vector lattice homomorphism for fixed $f_j \in E_j^+$ ($j \neq k$). Every vector lattice s -morphism is positive, and every positive $s_\mathbb{K}$ -linear map is real. At times, for emphasis, we will refer to a vector lattice s -morphism between Archimedean vector lattices over \mathbb{C} , (respectively, Archimedean vector lattices over \mathbb{R}) as a vector lattice $s_\mathbb{C}$ -morphism, or a vector lattice \mathbb{C} -homomorphism when $s = 1$ (respectively, a vector lattice $s_\mathbb{R}$ -morphism, or a vector lattice \mathbb{R} -homomorphism when $s = 1$).

Given an Archimedean vector lattice E over \mathbb{K} , we call a vector subspace L of E a *vector sublattice* of E if $|f| \in L$ for every $f \in L$. If L is a vector subspace of E and for every $0 < g \in E^+$ there exists $f \in L \cap E^+$ such that $0 < f \leq g$, we say that L is *order dense* in E . If a vector subspace L of E has the property that $f \in L, g \in E$ and $|g| \leq |f|$ imply that $g \in L$, we call L an *ideal* in E . Hence every ideal of E is a vector sublattice of E .

In [6], we develop the uniform completion for Archimedean vector lattices over \mathbb{R} in a manner that works just as well for developing the uniform completion for Archimedean vector lattices over \mathbb{C} . The definitions for relatively uniformly convergent sequences and relatively uniformly Cauchy sequences in an Archimedean vector lattice over \mathbb{K} , as well as the definition of a uniform completion of an Archimedean vector lattice over \mathbb{K} , are identical to what is found in Definitions 2.1 and 2.2 in [6], modulo replacing \mathbb{R} with \mathbb{K} . We also note that Propositions 3.1 and 3.2 in [6] also hold for Archimedean vector lattices over \mathbb{K} , and the proofs are identical to the real case. In particular, there exists an essentially unique uniform completion of every Archimedean vector lattice over \mathbb{K} .

Example 2.3. *If E is a uniformly complete Archimedean vector lattice over \mathbb{R} then $E_\mathbb{C}$ is a uniformly complete Archimedean vector lattice over \mathbb{C} . If E is a uniformly complete*

Archimedean vector lattice over \mathbb{C} then E_ρ is a uniformly complete Archimedean vector lattice over \mathbb{R} . Dedekind complete Archimedean vector lattices over \mathbb{K} are uniformly complete.

Let E be an Archimedean vector lattice over \mathbb{K} and let A be a subset of E . The *pseudo uniform closure* \bar{A} of A is the set of all $f \in E$ for which there exists a sequence (f_n) in A such that $f_n \xrightarrow{ru} f$ (also see page 85 of [14]). We call A *relatively uniformly closed* if $\bar{A} = A$, and we say that A is *uniformly dense* in E if $\bar{A} = E$.

Like in [6], we use transfinite induction to iterate the pseudo uniform closure of a vector sublattice L of an Archimedean vector lattice E over \mathbb{K} . For an Archimedean vector lattice E over \mathbb{K} and a nonempty subset A of E , we define

$$\begin{aligned} A_1 &:= A, \\ A_\alpha &:= \overline{A_{\alpha-1}} \text{ when } \alpha > 1 \text{ is not a limit ordinal, and} \\ A_\alpha &:= \bigcup_{\beta < \alpha} A_\beta \text{ when } \alpha \text{ is a limit ordinal.} \end{aligned}$$

Given an Archimedean vector lattice E over \mathbb{K} and a nonempty subset $A \subseteq E$, we know from the (identical) complex analogue of Proposition 3.1 in [6] that A_{ω_1} is uniformly closed in E . It is quite possible however that there exists an ordinal $\alpha < \omega_1$ such that A_α is relatively uniformly closed in E . Motivated by this observation, we define the *density number* $\tau(A, E)$ of A in E by $\tau(A, E) := \min\{\alpha : \bar{A}_\alpha = E\}$.

3. Vector Lattice Complexifications

We discuss the specific case of Proposition 3.17 and Theorem 3.18 in [6] that we will use to complexify Archimedean vector lattices over \mathbb{R} and multilinear maps over \mathbb{R} . Using the notation from Example 3.4 in [6], let $\mu_{2,4}(x, y) = \sqrt{\frac{|x|^2 + |y|^2}{2}}$ ($x, y \in \mathbb{R}$). By Corollary 3.9 in [6] and the content following Corollary 3.10 in [6], if E is an Archimedean vector lattice over \mathbb{R} and $f, g \in E$ then $\mu_{2,4}(f, g) = \frac{1}{\sqrt{2}}(f \boxplus g)$, where $f \boxplus g := \sup\{f \cos \theta + g \sin \theta : \theta \in [0, 2\pi]\}$, as defined by the authors of [3]. Therefore, from Mittelmeyer and Wolff's Theorem 2.2, a vector space $E + iE$ over \mathbb{C} is an Archimedean vector lattice over \mathbb{C} if and only if E is a $\mu_{2,4}$ -complete Archimedean vector lattice over \mathbb{R} . We refer to the $\mu_{2,4}$ -completion $(E^{\mu_{2,4}}, \phi)$ of E as the *square mean completion* of E . Noting that $\mu_{2,4}$ is absolutely invariant (defined following Theorem 3.15 in [6]), we summarize the newly found information regarding functional completions for this special case in the following corollary of Proposition 3.17 and Theorem 3.18 in [6].

Corollary 3.1. *If E is an Archimedean vector lattice over \mathbb{R} then there exists a unique square mean completion $(E^{\mu_{2,4}}, \phi)$ of E . Moreover, if E_1, \dots, E_s, F are Archimedean vector lattices over \mathbb{R} with square mean completions $(E_k^{\mu_{2,4}}, \phi_k)$ ($k \in \{1, \dots, s\}$) and F is square mean complete, then for every vector lattice s -morphism $T : \times_{k=1}^s E_k \rightarrow F$ there exists a unique vector lattice s -morphism $T^{\mu_{2,4}} : \times_{k=1}^s E_k^{\mu_{2,4}} \rightarrow F$ such that $T^{\mu_{2,4}}(\phi_1(f_1), \dots, \phi_s(f_s)) = T(f_1, \dots, f_s)$. Furthermore, if F is uniformly complete and $T : \times_{k=1}^s E_k \rightarrow F$ is a positive s -linear map then there exists a unique positive s -linear map $T^{\mu_{2,4}} : \times_{k=1}^s E_k^{\mu_{2,4}} \rightarrow F$ such that $T^{\mu_{2,4}}(\phi_1(f_1), \dots, \phi_s(f_s)) = T(f_1, \dots, f_s)$ for every $f_k \in E_k$ ($k \in \{1, \dots, s\}$). Here ϕ_k is the natural embedding of E_k into E_k^u .*

We now turn to complexifications of Archimedean vector lattices over \mathbb{R} .

Definition 3.2. *For an Archimedean vector lattice E over \mathbb{R} we define a pair $(E_{|\mathbb{C}|}, \phi)$ to be a vector lattice complexification of E if the following hold.*

- (1) $E_{|\mathbb{C}|}$ is an Archimedean vector lattice over \mathbb{C} .
- (2) $\phi : E \rightarrow (E_{|\mathbb{C}|})_\rho$ is an injective vector lattice \mathbb{R} -homomorphism.
- (3) For every Archimedean vector lattice F over \mathbb{C} as well as for every vector lattice \mathbb{R} -homomorphism $T : E \rightarrow F_\rho$, there exists a unique vector lattice \mathbb{C} -homomorphism $T_{|\mathbb{C}|} : E_{|\mathbb{C}|} \rightarrow F$ such that $T_{|\mathbb{C}|} \circ \phi = T$.

We next prove the existence and uniqueness of vector lattice complexifications.

Theorem 3.3. *If E is an Archimedean vector lattice over \mathbb{R} then there exists a vector lattice complexification of E , unique up to vector lattice isomorphism.*

Proof. Let E be an Archimedean vector lattice over \mathbb{R} . By Corollary 3.1, there exists a unique square mean completion $(E^{\mu_{2,4}}, \phi)$ of E . Define $E_{|\mathbb{C}|} := (E^{\mu_{2,4}})_\mathbb{C}$ and observe that $E_{|\mathbb{C}|}$ is an Archimedean vector lattice over \mathbb{C} and that $(E_{|\mathbb{C}|})_\rho = E^{\mu_{2,4}}$. Next, let F be an Archimedean vector lattice over \mathbb{C} and let $T : E \rightarrow F_\rho$ be a vector lattice \mathbb{R} -homomorphism. Since F_ρ is square mean complete, there exists a unique vector lattice \mathbb{R} -homomorphism $T^{\mu_{2,4}} : E^{\mu_{2,4}} \rightarrow F_\rho$ such that $T^{\mu_{2,4}} \circ \phi = T$. Define $T_{|\mathbb{C}|} : E_{|\mathbb{C}|} \rightarrow F$ by $T_{|\mathbb{C}|}(f + ig) = T^{\mu_{2,4}}(f) + iT^{\mu_{2,4}}(g)$ for every $f + ig \in E_{|\mathbb{C}|}$. Then $T_{|\mathbb{C}|} \circ \phi = T$. Moreover, for $f + ig \in E_{|\mathbb{C}|}$ we have from Corollary 3.13 in [6] (see also Proposition 3.4 of [2]) that

$$T_{|\mathbb{C}|}(|f + ig|) = T^{\mu_{2,4}}(f \boxplus g) = T^{\mu_{2,4}}(f) \boxplus T^{\mu_{2,4}}(g) = |T_{|\mathbb{C}|}(f + ig)|.$$

Thus $T_{|\mathbb{C}|}$ is a vector lattice \mathbb{C} -homomorphism and therefore $(E_{|\mathbb{C}|}, \phi)$ is a vector lattice complexification of E . Next, we prove the uniqueness. To this end, suppose $(E_{1|\mathbb{C}|}, \phi_1)$ and $(E_{2|\mathbb{C}|}, \phi_2)$ are vector lattice complexifications of E . Then $((E_{1|\mathbb{C}|})_\rho, \phi_1)$ and $((E_{2|\mathbb{C}|})_\rho, \phi_2)$

are square mean completions of E , and hence there exists a vector lattice isomorphism $\gamma : (E_{1|\mathbb{C}|})_\rho \rightarrow (E_{2|\mathbb{C}|})_\rho$. Similar to $T_{|\mathbb{C}|}$ above, the map $\gamma_{\mathbb{C}} : E_{1|\mathbb{C}|} \rightarrow E_{2|\mathbb{C}|}$ defined by $\gamma_{\mathbb{C}}(f + ig) = \gamma(f) + i\gamma(g)$ is a vector lattice \mathbb{C} -homomorphism. The bijectivity of $\gamma_{\mathbb{C}}$ is evident. \square

For the square mean completion $(E^{\mu_{2,4}}, \phi)$ of E , we will from now on identify E with $\phi(E)$. Using this identification, we complexify positive $s_{\mathbb{R}}$ -linear maps (respectively, vector lattice $s_{\mathbb{R}}$ -morphisms) to positive $s_{\mathbb{C}}$ -linear maps (respectively, vector lattice $s_{\mathbb{C}}$ -morphisms) as follows. Let E_1, \dots, E_s, F be Archimedean vector lattices over \mathbb{R} with F square mean complete, and let $T : \times_{k=1}^s E_k \rightarrow F$ be a vector lattice $s_{\mathbb{R}}$ -morphism. For $(f_0^1 + if_1^1, \dots, f_0^s + if_1^s) \in \times_{k=1}^s E_{k|\mathbb{C}|}$, define $T_{|\mathbb{C}|} : \times_{k=1}^s E_{k|\mathbb{C}|} \rightarrow F_{\mathbb{C}}$ by

$$T_{|\mathbb{C}|}(f_0^1 + if_1^1, \dots, f_0^s + if_1^s) := \sum_{\epsilon_k \in \{0,1\}} T^{\mu_{2,4}}(f_{\epsilon_1}^1, \dots, f_{\epsilon_s}^s) i^{\sum_{k=1}^s \epsilon_k}.$$

If F is uniformly complete and T above is any positive $s_{\mathbb{R}}$ -linear map, we define $T_{|\mathbb{C}|}$ in a similar manner. We collect a few facts regarding this complexification in the following proposition. Statement (3) and the statement that $T_{|\mathbb{C}|} = (T^{\mu_{2,4}})_{\mathbb{C}}$ in (1) and (2) are evident. The proof of (2) follows from Corollary 3.1, and the proof of (1) is similar to the complexification of vector lattice homomorphisms seen in the proof of Theorem 3.3.

Proposition 3.4. *Let E_1, \dots, E_s, F be Archimedean vector lattices over \mathbb{R} with F square mean complete.*

- (1) *If a map $T : \times_{k=1}^s E_k \rightarrow F$ is a vector lattice $s_{\mathbb{R}}$ -morphism then $T_{|\mathbb{C}|}$ is a vector lattice $s_{\mathbb{C}}$ -morphism and $T_{|\mathbb{C}|} = (T^{\mu_{2,4}})_{\mathbb{C}}$.*
- (2) *If F is uniformly complete and $T : \times_{k=1}^s E_k \rightarrow F$ is a positive $s_{\mathbb{R}}$ -linear map then $T_{|\mathbb{C}|}$ is a positive $s_{\mathbb{C}}$ -linear map and $T_{|\mathbb{C}|} = (T^{\mu_{2,4}})_{\mathbb{C}}$.*
- (3) *If in (1) or (2) all E_1, \dots, E_s are square mean complete then $T_{|\mathbb{C}|} = T_{\mathbb{C}}$.*

4. The Archimedean Vector Lattice Tensor Product

In this section, we define the tensor product of Archimedean vector lattices over \mathbb{K} and prove the existence of the Archimedean complex tensor product by complexifying the Fremlin tensor product of Archimedean real vector lattices.

We start with the definition of these tensor products of Archimedean vector lattices over \mathbb{K} . For $s = 2$ and $\mathbb{K} = \mathbb{R}$ the definition coincides with Fremlin's definition of the Archimedean tensor product of Archimedean vector lattices over \mathbb{R} (see [9]).

Definition 4.1. *Given Archimedean vector lattices E_1, \dots, E_s over \mathbb{K} , we define a pair $(\bar{\otimes}_{k=1}^s E_k, \bar{\otimes})$ to be an Archimedean vector lattice tensor product of E_1, \dots, E_s if the following hold.*

- (1) $\bar{\otimes}_{k=1}^s E_k$ is an Archimedean vector lattice over \mathbb{K} .
- (2) $\bar{\otimes}$ is a vector lattice s -morphism.
- (3) For every Archimedean vector lattice F over \mathbb{K} and for every vector lattice s -morphism $T : \times_{k=1}^s E_k \rightarrow F$, there exists a uniquely determined vector lattice homomorphism $T^{\bar{\otimes}} : \bar{\otimes}_{k=1}^s E_k \rightarrow F$ such that $T^{\bar{\otimes}} \circ \bar{\otimes} = T$.

Following its original proof, one can extend Fremlin's Theorem 4.2 in [9] to the vector lattice tensor product of any number of factors (see also [23], Section 2). Below and throughout the rest of this section, $(\otimes_{k=1}^s V_k, \otimes)$ denotes the algebraic tensor product of vector spaces V_1, \dots, V_s over \mathbb{K} .

Lemma 4.2. *Let E_1, \dots, E_s be Archimedean vector lattices over \mathbb{R} .*

- (1) *There exists an essentially unique Archimedean vector lattice $\bar{\otimes}_{k=1}^s E_k$ over \mathbb{R} and a vector lattice s -morphism $\bar{\otimes} : \times_{k=1}^s E_k \rightarrow \bar{\otimes}_{k=1}^s E_k$ such that for every Archimedean vector lattice F over \mathbb{R} and every vector lattice s -morphism $T : \times_{k=1}^s E_k \rightarrow F$, there exists a unique vector lattice homomorphism $T^{\bar{\otimes}} : \bar{\otimes}_{k=1}^s E_k \rightarrow F$ such that $T^{\bar{\otimes}} \circ \bar{\otimes} = T$.*
- (2) *There exists an injective linear map $S : \times_{k=1}^s E_k \rightarrow \bar{\otimes}_{k=1}^s E_k$ such that $S \circ \otimes = \bar{\otimes}$.*
- (3) *For every $w \in \bar{\otimes}_{k=1}^s E_k$, there exist $x_k \in E_k^+$ ($k \in \{1, \dots, s\}$) such that for every $\epsilon > 0$, there exists $v \in \otimes_{k=1}^s E_k$ such that $|w - v| \leq \epsilon(x_1 \otimes \dots \otimes x_s)$, i.e. $\otimes_{k=1}^s E_k$ is relatively uniformly dense in $\bar{\otimes}_{k=1}^s E_k$.*
- (4) *For every $0 < w \in \bar{\otimes}_{k=1}^s E_k$ there exist $x_k \in E_k^+$ ($k \in \{1, \dots, s\}$) such that $0 < (x_1 \otimes \dots \otimes x_s) \leq w$, i.e. $\otimes_{k=1}^s E_k$ is order dense in $\bar{\otimes}_{k=1}^s E_k$.*

The main result of this section deals with the existence and uniqueness of the complex Archimedean vector lattice tensor product and requires several prerequisite results. The next lemma surely is known but we could only find an explicit reference in the literature for a special case in the thesis [27].

Lemma 4.3. *If V_1, \dots, V_s are vector spaces over \mathbb{R} then $\otimes_{k=1}^s (V_k \mathbb{C})$ and $(\otimes_{k=1}^s V_k)_{\mathbb{C}}$ are isomorphic as vector spaces over \mathbb{C} .*

Proof. Since the algebraic tensor product is associative, we only need to prove the result for $s = 2$, and use induction. The case $s = 2$ is the content of Theorem 2.1.2 in [27], but, we provide a sketch of van Zyl's proof to correct some potential confusion caused by an accumulation of minor misprints. First let U and V be vector spaces over \mathbb{R} , and let $(U \otimes V, \otimes)$ and $(U_{\mathbb{C}} \otimes_1 V_{\mathbb{C}})$ be the algebraic tensor products of U, V , respectively $U_{\mathbb{C}}, V_{\mathbb{C}}$. Since $\otimes_{\mathbb{C}} : U_{\mathbb{C}} \times V_{\mathbb{C}} \rightarrow (U \otimes V)_{\mathbb{C}}$ is a bilinear map over \mathbb{C} , it induces a unique \mathbb{C} -linear map $T : U_{\mathbb{C}} \otimes_1 V_{\mathbb{C}} \rightarrow (U \otimes V)_{\mathbb{C}}$. It is easy to see that T is surjective. To show that T is injective, let $w = \sum_{k=1}^n (u_k + iu'_k) \otimes_1 (v_k + iv'_k) \in U_{\mathbb{C}} \otimes_1 V_{\mathbb{C}}$ and suppose that $T(w) = 0$. Note that $T(w) = \sum_{k=1}^n (u_k \otimes v_k - u'_k \otimes v'_k + iu'_k \otimes v_k + iu_k \otimes v'_k)$, and so for any \mathbb{R} -linear functionals ϕ on U and ψ on V we have

$$\sum_{k=1}^n (\phi(u_k)\psi(v_k) - \phi(u'_k)\psi(v'_k)) = 0 \text{ and } \sum_{k=1}^n (\phi(u'_k)\psi(v_k) + \phi(u_k)\psi(v'_k)) = 0. \quad (*)$$

Let $\xi = \xi_r + i\xi_c$ be a \mathbb{C} -linear functional on $U_{\mathbb{C}}$ and let $\eta = \eta_r + i\eta_c$ be a \mathbb{C} -linear functional on $V_{\mathbb{C}}$, both written in their natural decompositions. Then ξ_r, ξ_c are \mathbb{R} -linear functionals on U and η_r, η_c are \mathbb{R} -linear functionals on V . Now

$$\begin{aligned} & \sum_{k=1}^n \xi(u_k + iu'_k)\eta(v_k + iv'_k) \\ &= \sum_{k=1}^n (\xi_r(u_k)\eta_r(v_k) - \xi_r(u'_k)\eta_r(v'_k)) - \sum_{k=1}^n (\xi_r(u'_k)\eta_c(v_k) + \xi_r(u_k)\eta_c(v'_k)) \\ &+ i \sum_{k=1}^n (\xi_r(u'_k)\eta_r(v_k) + \xi_r(u_k)\eta_r(v'_k)) + i \sum_{k=1}^n (\xi_r(u_k)\eta_c(v_k) - \xi_r(u'_k)\eta_c(v'_k)) \\ &- \sum_{k=1}^n (\xi_c(u'_k)\eta_r(v_k) + \xi_c(u_k)\eta_r(v'_k)) - \sum_{k=1}^n (\xi_c(u_k)\eta_c(v_k) - \xi_c(u'_k)\eta_c(v'_k)) \\ &+ i \sum_{k=1}^n (\xi_c(u_k)\eta_r(v_k) - \xi_c(u'_k)\eta_r(v'_k)) - i \sum_{k=1}^n (\xi_c(u'_k)\eta_c(v_k) + \xi_c(u_k)\eta_c(v'_k)). \end{aligned}$$

Applying $(*)$ again to each of these eight summands, we have that $\sum_{k=1}^n \xi(u_k + iu'_k)\eta(v_k + iv'_k) = 0$. Therefore $w = 0$ and T is injective. Then T is a vector space isomorphism. \square

In light of the previous lemma, we will from now identify $(\otimes_{k=1}^s V_k \mathbb{C})_{\rho}$ with $\otimes_{k=1}^s V_k$ for vector spaces V_1, \dots, V_s over \mathbb{R} .

Next, we note that there exists a simpler construction of the square mean completion than the construction preceding Proposition 3.16 in [6], which was given in a more general setting. Indeed, in Remark 4 of [2], Azouzi constructs a square mean completion of an Archimedean vector lattice E over \mathbb{R} essentially as follows. Let $E_1 := E$ and for every $n \in \mathbb{N}$, define $E_{n+1} := E_n \cup [\{\mu_{2,4}(f, g) : f, g \in E_n\}]$, where $[\{\mu_{2,4}(f, g) : f, g \in E_n\}]$ denotes the vector subspace of E^δ generated by $\{\mu_{2,4}(f, g) : f, g \in E_n\}$. Then define $E^\boxplus := \bigcup_{n \in \mathbb{N}} E_n$. To see that E^\boxplus is a vector lattice, note that for every $f \in E^\boxplus$ there exists $n \in \mathbb{N}$ such that $f \in E_n$. Then $|f| = \sqrt{2}\mu_{2,4}(f, 0) \in E_{n+1}$. It follows that E^\boxplus is the square mean completion of E , that is, E^\boxplus and $E^{\mu_{2,4}}$ are isomorphic as vector lattices. In fact, from the identity $\lambda\mu_{2,4}(f, g) = \mu_{2,4}(\lambda f, \lambda g)$ for every $\lambda \in \mathbb{R}^+$ and every $f, g \in E$, we have $E_{n+1}^+ = \{\sum_{k=1}^m \mu_{2,4}(f_k, g_k) : f_k, g_k \in E_n\}$. We use this fact in the first of the two following lemmas that are needed for Proposition 4.6.

Lemma 4.4. *Denote the standard sine and cosine functions on $[0, \frac{\pi}{2}]$ by \sin and \cos , respectively. For an Archimedean vector lattice E over \mathbb{R} and for every $f \in (E^{\mu_{2,4}})^+$ there exists $u_1, \dots, u_n \in E^+$ and $t_{k,1}, \dots, t_{k,p_k} \in \{\cos, \sin\}$ ($k \in \{1, \dots, n\}$) such that*

$$f = \sup_{\theta_k \in [0, \frac{\pi}{2}]} \left\{ \sum_{k=1}^n \prod_{j=1}^{p_k} t_{k,j}(\theta_k) u_k \right\}.$$

Proof. Our proof is via mathematical induction. Let $h \in E_2^+$ and first suppose that $f = \mu_{2,4}(u, v)$ for some $u, v \in E^+$. Then $f = \sup\{u \cos \theta + v \sin \theta : \theta \in [0, \frac{\pi}{2}]\}$. Next, suppose that $f = \sum_{k=1}^n \mu_{2,4}(u_k, v_k)$. Then

$$f = \sum_{k=1}^n \sup_{\theta_k \in [0, \frac{\pi}{2}]} \{u_k \cos \theta_k + v_k \sin \theta_k\} = \sup_{\theta_k \in [0, \frac{\pi}{2}]} \left\{ \sum_{k=1}^n (u_k \cos \theta_k + v_k \sin \theta_k) \right\}.$$

This completes the base step of the induction argument. For the inductive step, suppose that for every $f \in E_n^+$ there exists $u_1, \dots, u_n \in E^+$ and $t_1, \dots, t_{p_k} \in \{\cos, \sin\}$ ($k \in \{1, \dots, n\}$) such that

$$f = \sup_{\theta_{k,j} \in [0, \frac{\pi}{2}]} \left\{ \sum_{k=1}^n \prod_{j=1}^{p_k} t_{k,j}(\theta_{k,j}) u_k \right\}.$$

Let $f \in E_{n+1}^+$. From the argument in the base step above, we may assume that $f = \mu_{2,4}(u, v)$ for some $u, v \in E_n^+$. By the induction hypothesis there exists $u_1, \dots, u_n, v_1, \dots, v_n \in E^+$ and $t_{k,1}, \dots, t_{k,p_k}, s_{k,1}, \dots, s_{k,r_k} \in \{\cos, \sin\}$ ($k \in \{1, \dots, n\}$) such that

$$u = \sup_{\theta_{k,j} \in [0, \frac{\pi}{2}]} \left\{ \sum_{k=1}^n \prod_{j=1}^{p_k} t_{k,j}(\theta_{k,j}) u_k \right\} \text{ and } v = \sup_{\theta_{k,j} \in [0, \frac{\pi}{2}]} \left\{ \sum_{k=1}^n \prod_{j=1}^{r_k} s_{k,j}(\theta_{k,j}) v_k \right\}.$$

Then

$$\begin{aligned}
f &= \mu_{2,4} \left(\sup_{\theta_{k,j} \in [0, \frac{\pi}{2}]} \left\{ \sum_{k=1}^n \prod_{j=1}^{p_k} t_{k,j}(\theta_{k,j}) u_k \right\}, \sup_{\theta_{k,j} \in [0, \frac{\pi}{2}]} \left\{ \sum_{k=1}^m \prod_{j=1}^{r_k} s_{k,j}(\theta_{k,j}) v_k \right\} \right) \\
&= \sup_{\phi \in [0, \frac{\pi}{2}]} \left\{ \sup_{\theta_{k,j} \in [0, \frac{\pi}{2}]} \left\{ \sum_{k=1}^n \prod_{j=1}^{p_k} t_{k,j}(\theta_{k,j}) u_k \right\} \cos \phi + \sup_{\theta_{k,j} \in [0, \frac{\pi}{2}]} \left\{ \sum_{k=1}^m \prod_{j=1}^{r_k} s_{k,j}(\theta_{k,j}) v_k \right\} \sin \phi \right\} \\
&= \sup_{\phi, \theta_{j,k} \in [0, \frac{\pi}{2}]} \left\{ \sum_{k=1}^n \prod_{j=1}^{p_k} t_{k,j}(\theta_{k,j}) \cos \phi u_k + \sum_{k=1}^m \prod_{j=1}^{r_k} s_{k,j}(\theta_{k,j}) \sin \phi v_k \right\}.
\end{aligned}$$

□

The next lemma can be verified using mathematical induction. We do not include the proof.

Lemma 4.5. *Let t_1, \dots, t_n be Lipschitz functions on \mathbb{R} with Lipschitz constant 1. Also assume that $|t_k(x)| \leq 1$ for every $k \in \{1, \dots, n\}$ and every $x \in \mathbb{R}$. Then for every $x_k, y_k \in \mathbb{R}$ ($k \in \{1, \dots, n\}$) we have $\left| \prod_{k=1}^n t_k(x_k) - \prod_{k=1}^n t_k(y_k) \right| \leq \sum_{k=1}^n |x_k - y_k|$.*

The idea of the proof for the following proposition comes from Lemma 2.8 in [2].

Proposition 4.6. *If E is an Archimedean vector lattice over \mathbb{R} then E is relatively uniformly dense in $E^{\mu_{2,4}}$.*

Proof. Let E be an Archimedean vector lattice over \mathbb{R} and first suppose that $f \in (E^{\mu_{2,4}})^+$. Say that $f = \sup_{\theta_{k,j} \in [0, \frac{\pi}{2}]} \left\{ \sum_{k=1}^n \prod_{j=1}^{p_k} t_{k,j}(\theta_{k,j}) u_k \right\}$ for some $u_1, \dots, u_n \in E^+$ and $t_{k,1}, \dots, t_{k,p_k} \in \{\cos, \sin\}$ ($k \in \{1, \dots, n\}$). Note that given $\theta_{k,j} \in [0, \frac{\pi}{2}]$ $m \in \mathbb{N}$ there exist $l_{k,j} \in \mathbb{N}$ such that $|\frac{l_{k,j}\pi}{2^m} - \theta_{k,j}| \leq \frac{\pi}{2^m}$. Since sine and cosine are both Lipschitz functions with Lipschitz constant 1 we have from Lemma 4.5 that

$$\begin{aligned}
\left| \sum_{k=1}^n \prod_{j=1}^{p_k} t_{k,j}(\theta_{k,j}) u_k - \sum_{k=1}^n \prod_{j=1}^{p_k} t_{k,j}(\frac{l_{k,j}\pi}{2^m}) u_k \right| &\leq \sum_{k=1}^n \left| \prod_{j=1}^{p_k} t_{k,j}(\theta_{k,j}) - \prod_{j=1}^{p_k} t_{k,j}(\frac{l_{k,j}\pi}{2^m}) \right| |u_k| \\
&\leq \sum_{k=1}^n \sum_{j=1}^{p_k} |\theta_{k,j} - \frac{l_{k,j}\pi}{2^m}| |u_k| \\
&\leq \frac{\pi}{2^m} \sum_{k=1}^n p_k |u_k|.
\end{aligned}$$

Thus,

$$\begin{aligned} \sum_{k=1}^n \prod_{j=1}^{p_k} t_{k,j}(\theta_{k,j}) u_k &\leq \sum_{k=1}^n \prod_{j=1}^{p_k} t_{k,j}\left(\frac{l_{k,j}\pi}{2^m}\right) u_k + \frac{\pi}{2^m} \sum_{k=1}^n p_k |u_k| \\ &\leq \bigvee_{l_{k,j}=1}^{2^m} \sum_{k=1}^n \prod_{j=1}^{p_k} t_{k,j}\left(\frac{l_{k,j}\pi}{2^m}\right) u_k + \frac{\pi}{2^m} \sum_{k=1}^n p_k |u_k|. \end{aligned}$$

Since this is true for every $\theta_{k,j} \in [0, \frac{\pi}{2}]$ ($k \in \{1, \dots, n\}, j \in \{1, \dots, p_k\}$) we have

$$0 \leq f - \bigvee_{l_{k,j}=1}^{2^m} \sum_{k=1}^n \prod_{j=1}^{p_k} t_{k,j}\left(\frac{l_{k,j}\pi}{2^m}\right) \leq \frac{\pi}{2^m} \sum_{k=1}^n p_k |u_k|.$$

It follows that the sequence $\sigma_m := \bigvee_{l_{k,j}=1}^{2^m} \sum_{k=1}^n \prod_{j=1}^{p_k} t_{k,j}\left(\frac{l_{k,j}\pi}{2^m}\right)$ converges relatively uniformly to f .

Finally, for $f \in E$, there exist sequences $(a_n), (b_n)$ in E such that $a_n \xrightarrow{ru} f^+$ and $b_n \xrightarrow{ru} f^-$. Then $a_n - b_n \xrightarrow{ru} f$. \square

We are ready to deal with the Archimedean tensor product of Archimedean vector lattices over \mathbb{K} . Parts (1), (2), and (4) of the following theorem extend the corresponding parts of Theorem 4.2 in [9] and Lemma 4.2. Parts (2) and (4) generalize corresponding results by Schep for real Archimedean vector lattices in Section 2 of [23]. Part (3) is slightly weaker than the real analogues found in [9] and [23], but it is sufficient for obtaining our results for maps of order bounded variation in the next section. We do not yet know if it is possible to strengthen (3).

Theorem 4.7. *Let E_1, \dots, E_s be Archimedean vector lattices over \mathbb{K} .*

- (1) *There exists an essentially unique Archimedean vector lattice $\bar{\otimes}_{k=1}^s E_k$ over \mathbb{K} and a vector lattice s -morphism $\bar{\otimes} : \times_{k=1}^s E_k \rightarrow \bar{\otimes}_{k=1}^s E_k$ such that for every Archimedean vector lattice F over \mathbb{K} and every vector lattice s -morphism $T : \times_{k=1}^s E_k \rightarrow F$, there exists a unique vector lattice homomorphism $T^{\bar{\otimes}} : \bar{\otimes}_{k=1}^s E_k \rightarrow F$ such that $T^{\bar{\otimes}} \circ \bar{\otimes} = T$.*
- (2) *There exists an injective \mathbb{K} -linear map $S : \otimes_{k=1}^s E_k \rightarrow \bar{\otimes}_{k=1}^s E_k$ such that $S \circ \otimes = \bar{\otimes}$.*
- (3) *$\tau(\otimes_{k=1}^s E_k, \bar{\otimes}_{k=1}^s E_k) \leq 2$. Thus, $\otimes_{k=1}^s E_k$ is dense in $\bar{\otimes}_{k=1}^s E_k$ in the relatively uniform topology.*
- (4) *For every $w \in (\bar{\otimes}_{k=1}^s E_k) \setminus \{0\}$ there exist $x_1 \otimes \dots \otimes x_s \in \otimes_{k=1}^s E_{k\rho}$ with $x_k \in E_k^+$ ($k \in \{1, \dots, s\}$) such that $0 < (x_1 \otimes \dots \otimes x_s) \leq |w|$, i.e. $\otimes_{k=1}^s E_k$ is order dense in $\bar{\otimes}_{k=1}^s E_k$.*

Proof. By Lemma 4.2, statements (1)-(4) are valid for $\mathbb{K} = \mathbb{R}$. We assume in the proof below that $\mathbb{K} = \mathbb{C}$.

(1) Let E_1, \dots, E_s, F be Archimedean vector lattices over \mathbb{C} . Denote by $(\bar{\otimes}_{k=1}^s E_{k\rho}, \bar{\otimes})$ the Archimedean vector lattice tensor product of $E_{1\rho}, \dots, E_{s\rho}$. We claim that the pair $((\bar{\otimes}_{k=1}^s E_{k\rho})_{|\mathbb{C}|}, \bar{\otimes}_{|\mathbb{C}|})$ is the unique Archimedean complex vector lattice tensor product of E_1, \dots, E_s . Let $T : \times_{k=1}^s E_k \rightarrow F$ be a vector lattice s -morphism. From Lemma 4.2, the map $\bar{\otimes}$ induces a unique vector lattice homomorphism $T_\rho^{\bar{\otimes}}$ on $\bar{\otimes}_{k=1}^s E_{k\rho}$ such that $T_\rho^{\bar{\otimes}} \circ \bar{\otimes} = T_\rho$. Also, the map $T_\rho^{\bar{\otimes}}$ extends uniquely to a vector lattice homomorphism $(T_\rho^{\bar{\otimes}})^{\mu_{2,4}}$ on $(\bar{\otimes}_{k=1}^s E_{k\rho})^{\mu_{2,4}}$ (Corollary 3.1). By Proposition 3.4(1), the map $\bar{\otimes}_{|\mathbb{C}|}$ is a vector lattice s -morphism and $(T_\rho^{\bar{\otimes}})_{|\mathbb{C}|}$ is a vector lattice homomorphism. We will prove that the map $(T_\rho^{\bar{\otimes}})_{|\mathbb{C}|} : (\bar{\otimes}_{k=1}^s E_{k\rho})_{|\mathbb{C}|} \rightarrow F$ is the unique vector lattice homomorphism such that $(T_\rho^{\bar{\otimes}})_{|\mathbb{C}|} \circ \bar{\otimes}_{|\mathbb{C}|} = T$. Indeed, for every $(f_0^1 + if_1^1, \dots, f_0^s + if_1^s) \in \times_{k=1}^s E_k$ we have

$$\begin{aligned}
(T_\rho^{\bar{\otimes}})_{|\mathbb{C}|} \circ \bar{\otimes}_{|\mathbb{C}|}(f_0^1 + if_1^1, \dots, f_0^s + if_1^s) &= (T_\rho^{\bar{\otimes}})_{|\mathbb{C}|} \left(\sum_{\epsilon_k \in \{0,1\}} \bar{\otimes}(f_{\epsilon_1}^1, \dots, f_{\epsilon_s}^s) i^{\sum_{k=1}^s \epsilon_k} \right) \\
&= \sum_{\epsilon_k \in \{0,1\}} T_\rho^{\bar{\otimes}} \circ \bar{\otimes}(f_{\epsilon_1}^1, \dots, f_{\epsilon_s}^s) i^{\sum_{k=1}^s \epsilon_k} \\
&= \sum_{\epsilon_k \in \{0,1\}} T_\rho(f_{\epsilon_1}^1, \dots, f_{\epsilon_s}^s) i^{\sum_{k=1}^s \epsilon_k} \\
&= T(f_0^1 + if_1^1, \dots, f_0^s + if_1^s).
\end{aligned}$$

Since every vector lattice \mathbb{C} -homomorphism is real, the uniqueness of $(T_\rho^{\bar{\otimes}})_{|\mathbb{C}|}$ follows from the uniqueness of $(T_\rho^{\bar{\otimes}})^{\mu_{2,4}}$.

The proof of uniqueness of the Archimedean complex vector lattice tensor product is not different from the real case.

(2) Consider the newly minted tensor product $(\bar{\otimes}_{k=1}^s E_k, \bar{\otimes})$ constructed in (1). By Lemma 4.2, there exists an Archimedean vector lattice G over \mathbb{R} and a vector lattice s -morphism $T : \times_{k=1}^s E_{k\rho} \rightarrow G$ such that the induced linear map $T^\otimes : \otimes_{k=1}^s E_{k\rho} \rightarrow G$ is injective. By taking the square mean completion of G , if necessary, we will assume that G is square mean complete. By taking vector space complexifications, we find an injective vector lattice s -morphism $(T^\otimes)_\mathbb{C} : (\otimes_{k=1}^s E_{k\rho})_\mathbb{C} \rightarrow G_\mathbb{C}$, or equivalently by Lemma 4.3, $(T^\otimes)_\mathbb{C} : \otimes_{k=1}^s E_k \rightarrow G_\mathbb{C}$. Moreover, if $(T_\mathbb{C})^\otimes : \otimes_{k=1}^s E_k \rightarrow G$ is the unique linear map induced

by $T_{\mathbb{C}}$, then for $f_0^k + if_1^k \in E_k$ ($k \in \{1, \dots, s\}$),

$$\begin{aligned} (T_{\mathbb{C}})^{\otimes}(f_0^1 + if_1^1 \otimes \dots \otimes f_0^s + if_1^s) &= T_{\mathbb{C}}(f_0^1 + if_1^1, \dots, f_0^s + if_1^s) \\ &= \sum_{\epsilon_k \in \{0,1\}} T(f_{\epsilon_1}^1, \dots, f_{\epsilon_s}^s) i_{k=1}^{\sum \epsilon_k} \\ &= \sum_{\epsilon_k \in \{0,1\}} T^{\otimes}(f_{\epsilon_1}^1 \otimes \dots \otimes f_{\epsilon_s}^s) i_{k=1}^{\sum \epsilon_k}. \end{aligned}$$

In particular, $((T_{\mathbb{C}})^{\otimes})$ is a real map with $((T_{\mathbb{C}})^{\otimes})_{\rho} = T^{\otimes}$, and therefore we have $(T_{\mathbb{C}})^{\otimes} = (T^{\otimes})_{\mathbb{C}}$. From part (1) of this theorem there exists a unique vector lattice \mathbb{C} -homomorphism $(T_{\mathbb{C}})^{\bar{\otimes}} : \bar{\otimes}_{k=1}^s E_k \rightarrow G_{\mathbb{C}}$ such that $(T_{\mathbb{C}})^{\bar{\otimes}} \circ \bar{\otimes} = T_{\mathbb{C}}$. Moreover, there exists a unique \mathbb{C} -linear map $S : \bar{\otimes}_{k=1}^s E_k \rightarrow \bar{\otimes}_{k=1}^s E_k$ such that $S \circ \bar{\otimes} = \bar{\otimes}$. Then $(T_{\mathbb{C}})^{\bar{\otimes}} \circ S \circ \bar{\otimes} = T_{\mathbb{C}}$, and hence $(T_{\mathbb{C}})^{\bar{\otimes}} \circ S = (T_{\mathbb{C}})^{\otimes} = (T^{\otimes})_{\mathbb{C}}$. Therefore S is injective.

(3) By Lemma 4.2 we know that $\bar{\otimes}_{k=1}^s E_{k\rho}$ is relatively uniformly dense in $\bar{\otimes}_{k=1}^s E_{k\rho}$. We also know from Proposition 4.6 that $\bar{\otimes}_{k=1}^s E_{k\rho}$ is relatively uniformly dense in $(\bar{\otimes}_{k=1}^s E_{k\rho})^{\mu_{2,4}}$. By taking vector space complexifications, we have $\tau(\bar{\otimes}_{k=1}^s E_k, \bar{\otimes}_{k=1}^s E_k) \leq 2$.

(4) Suppose $w \in (\bar{\otimes}_{k=1}^s E_k) \setminus \{0\}$. Then $0 < |w| \in (\bar{\otimes}_{k=1}^s E_{k\rho})^{\mu_{2,4}}$. Since $\bar{\otimes}_{k=1}^s E_{k\rho}$ is order dense in $(\bar{\otimes}_{k=1}^s E_{k\rho})^{\delta}$, it is also order dense in $(\bar{\otimes}_{k=1}^s E_{k\rho})^{\mu_{2,4}}$. Thus there exists $w_0 \in \bar{\otimes}_{k=1}^s E_{k\rho}$ such that $0 < w_0 \leq |w|$. From Lemma 4.2, there exists $x_1 \otimes \dots \otimes x_s \in \bar{\otimes}_{k=1}^s E_{k\rho}$ with $x_k \in E_k^+$ ($k \in \{1, \dots, s\}$) such that $0 < (x_1 \otimes \dots \otimes x_s) \leq w_0$. \square

In (1) it is necessary to take the vector lattice complexification of $\bar{\otimes}_{k=1}^s E_{k\rho}$ to ensure that $(\bar{\otimes}_{k=1}^s E_{k\rho})_{|\mathbb{C}|}$ is an Archimedean vector lattice over \mathbb{C} . Indeed, Theorems 4.10 and 4.11 furnish examples where the vector space complexification $(\bar{\otimes}_{k=1}^s E_{k\rho})_{\mathbb{C}}$ does not suffice. We need two lemmas first.

Lemma 4.8. *Let X and Y be nonempty subsets of \mathbb{R} without isolated points. Then the function $S : (x, y) \mapsto \sqrt{x^2 + y^2}$ ($(x, y) \in X \times Y$) is in the square mean completion of $C(X) \bar{\otimes} C(Y)$ but for all nonempty open subsets U of X and W of Y we have $S|_{U \times W} \notin C(U) \otimes C(W)$.*

Proof. For $f \in C(X)$ and $g \in C(Y)$ we identify $f \otimes g$ with the function $(x, y) \mapsto f(x)g(y)$ ($(x, y) \in X \times Y$). Consider the element S of the square mean completion of $C(X) \bar{\otimes} C(Y)$ defined by

$$(x, y) \mapsto \sqrt{x^2 + y^2} \quad ((x, y) \in X \times Y).$$

Let U and W be open nonempty subsets of X and Y , respectively. We will show that the vector subspace of $C(U)$ generated by $\{S(\cdot, y) : y \in W\}$, whose elements are considered

as functions on U , is not finite-dimensional. It follows (see Proposition 1 in [10]) that $S|_{U \times W} \notin C(U) \otimes C(W)$. Since W is open and nonempty and Y has no isolated points, we can choose $\alpha_k \in W$ (for all $k \in \mathbb{N}$) for which $\alpha_i^2 \neq \alpha_j^2$ when $i \neq j$. Let $n \in \mathbb{N}$ and let $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ for which $\lambda_k \sqrt{x^2 + \alpha_k^2} = \lambda_k S(x, y_k) = 0$ for all $x \in U$. Since the function $x \mapsto \lambda_k \sqrt{x^2 + \alpha_k^2}$ ($x \in \mathbb{R}$) is n times differentiable at every $x \in X$, a routine calculation shows that the $n \times n$ matrix $A(x)$ defined by $A(x)_{ij} = \frac{1}{(x^2 + \alpha_j^2)^{\frac{2i-1}{2}}}$ when evaluated at the vector $(\lambda_1, \dots, \lambda_n)$ yields the vector $(0, \dots, 0)$ for every non-zero $x \in U$. However, $\prod_{k=1}^n \sqrt{x^2 + \alpha_k^2} \det(A(x)) = \det(B(x))$ where the $n \times n$ matrix $B(x)$ is defined by $B(x)_{ij} = \frac{1}{(x^2 + \alpha_j^2)^{i-1}}$, which has (Vandermonde) determinant

$$\prod_{1 \leq j < k \leq n} \left(\frac{1}{x^2 + \alpha_j^2} - \frac{1}{x^2 + \alpha_k^2} \right) = \prod_{1 \leq j < k \leq n} \frac{\alpha_j^2 - \alpha_k^2}{(x^2 + \alpha_j^2)(x^2 + \alpha_k^2)} \neq 0.$$

Thus $\det(A(x)) \neq 0$ for every non-zero $x \in U$, the vector subspace of $C(U)$ generated by $\{S(\cdot, y) : y \in Y\}$ (as functions on U) is infinite dimensional, and $S|_{U \times W} \notin C(U) \otimes C(W)$. \square

Lemma 4.9. *Let X and Y be nonempty subsets of \mathbb{R} without isolated points and let $f \in C(X) \bar{\otimes} C(Y)$. Then there exists a nonempty open subset V of $X \times Y$ and $g \in C(X) \otimes C(Y)$ such that $f|_V = g|_V$.*

Proof. Note that $C(X) \bar{\otimes} C(Y)$ is the vector lattice generated by $C(X) \otimes C(Y)$ in $C(X \times Y)$ ([9], Section 4). Every element $f \in C(X) \bar{\otimes} C(Y)$ is of the form $f = \bigwedge_{j=1}^n \bigvee_{k=1}^m f_{j,k}$ where $f_{j,k} \in C(X) \otimes C(Y)$ for each j and each k (see Exercise 4.8 in [1]). Let $f_1, f_2 \in C(X) \otimes C(Y)$. If $f_1 \neq f_2$, we may assume there exists (x, y) such that $f_1(x, y) < f_2(x, y)$ and then there exists a nonempty open subset O of $X \times Y$ such that $f_1 \wedge f_2 = f_1$ on O . Of course such an open subset O also exists if $f_1 = f_2$. By repeating this argument there exists a nonempty open set $U \subseteq O$ such that $\bigwedge_{j=1}^n \left(\bigvee_{k=1}^m f_{j,k} \right) = \bigvee_{k=1}^m f_{j_0,k}$ on U for some $j_0 \in \{1, \dots, n\}$. Similarly, there exists a nonempty open set $V \subset U$ such that $\bigvee_{k=1}^m f_{j_0,k_0} = f_{j_0,k_0}$ on V for some $k_0 \in \{1, \dots, m\}$. \square

Theorem 4.10. *Let X and Y be nonempty subsets of \mathbb{R} without isolated points. Then $C(X) \bar{\otimes} C(Y)$ is not square mean complete. Therefore $(C(X) \bar{\otimes} C(Y))_{\mathbb{C}}$ is not an Archimedean vector lattice over \mathbb{C} .*

Proof. Assume that the element S of Lemma 4.8 is in $C(X) \bar{\otimes} C(Y)$. Then by Lemma 4.9 there exists a nonempty open set V in $X \times Y$ and an element $g \in C(X) \otimes C(Y)$ such that $g|_V = S|_V$. However the open set V contains a nonempty open subset of the form $U \times W$ with $0 \notin U$. This contradicts Lemma 4.8. \square

We use Theorem 4.10 to prove the following.

Theorem 4.11. *If X and Y are uncountable compact metrizable spaces then $C(X) \bar{\otimes} C(Y)$ is not square mean complete. Therefore $(C(X) \bar{\otimes} C(Y))_{\mathbb{C}}$ is not an Archimedean vector lattice over \mathbb{C} .*

Proof. By Theorem 1 in [18], we know that both X and Y contain a closed subset homeomorphic with the Cantor set \mathbb{D} . Then $\mathbb{D} \times \mathbb{D}$ can be viewed as a closed subset of $X \times Y$ and the function $F_0 : (x, y) \mapsto \sqrt{x^2 + y^2} ((x, y) \in \mathbb{D} \times \mathbb{D})$ is continuous. By Tietze's Extension Theorem, the function $x \mapsto x$ ($x \in \mathbb{D}$) can be extended to continuous functions f and g on X and Y , respectively. Then the function $F : (x, y) \mapsto \sqrt{f(x)^2 + g(y)^2} ((x, y) \in X \times Y)$ is a continuous function in the square mean completion of $C(X) \bar{\otimes} C(Y)$ that extends F_0 . If F were in $C(X) \bar{\otimes} C(Y)$ itself then its restriction to $\mathbb{D} \times \mathbb{D}$ would be in $C(\mathbb{D}) \bar{\otimes} C(\mathbb{D})$ which by Theorem 4.10 is impossible. This proves the theorem. \square

It is certainly tempting to conjecture the following.

Conjecture 4.12. *If X and Y are infinite compact metrizable spaces then $C(X) \bar{\otimes} C(Y)$ is not square mean complete.*

The above two theorems show that the old way of complexifying Archimedean vector lattices via vector space complexifications is inadequate for pursuing complex analysis on Archimedean complex vector lattices.

We remark that the complex Archimedean vector lattice tensor product, like its real counterpart ([9], Theorem 5.3, [23], Section 2), possesses as well a universal property with respect to positive multilinear maps and complex uniformly complete vector lattices as range. The proof of this universal property, stated in the theorem below, is similar to the proof of Theorem 4.7(1) and is left to reader.

Theorem 4.13. *Let E_1, \dots, E_s, F be Archimedean vector lattices over \mathbb{K} with F uniformly complete. If $T : \times_{k=1}^s E_k \rightarrow F$ is a positive $s_{\mathbb{K}}$ -linear map, then there exists a unique positive \mathbb{K} -linear map $T^{\bar{\otimes}} : \bar{\otimes}_{k=1}^s E_k \rightarrow F$ such that $T^{\bar{\otimes}} \circ \bar{\otimes} = T$.*

A reformulation of part (1) of Theorem 4.7 in terms of Archimedean real vector lattices and vector lattice complexifications is the following.

Theorem 4.14. *Let E_1, \dots, E_s, F be Archimedean vector lattices over \mathbb{R} and suppose that $T : \times_{k=1}^s E_k \rightarrow F$ is a vector lattice $s_{\mathbb{R}}$ -morphism. There exists a unique vector lattice $s_{\mathbb{C}}$ -morphism $(T|_{\mathbb{C}})^{\bar{\otimes}} : \bar{\otimes}_{k=1}^s E_{k|_{\mathbb{C}}} \rightarrow F|_{\mathbb{C}}$ such that $(T|_{\mathbb{C}})^{\bar{\otimes}} \circ \bar{\otimes}|_{\times_{k=1}^s E_k} = T$.*

Proof. Consider T to be a vector lattice $s_{\mathbb{R}}$ -morphism from $\times_{k=1}^s E_k$ to $F^{\mu_2, 4}$. By Proposition 3.4(1) there exists a unique vector lattice $s_{\mathbb{C}}$ -morphism $T|_{\mathbb{C}} : \times_{k=1}^s E_{k|_{\mathbb{C}}} \rightarrow F|_{\mathbb{C}}$ such that $T|_{\mathbb{C}}|_{\times_{k=1}^s E_k} = T$. If $(T|_{\mathbb{C}})^{\bar{\otimes}}$ is the unique vector lattice \mathbb{C} -homomorphism induced by $T|_{\mathbb{C}}$ then $(T|_{\mathbb{C}})^{\bar{\otimes}} \circ \bar{\otimes} = T|_{\mathbb{C}}$. In particular, $(T|_{\mathbb{C}})^{\bar{\otimes}} \circ \bar{\otimes}|_{\times_{k=1}^s E_k} = T$. \square

5. Applications of the Archimedean Vector Lattice Tensor Product

In this section, we give some applications of the Archimedean vector lattice tensor product of Archimedean vector lattices over \mathbb{K} . Indeed, we use this tensor product to prove the existence of s -powers for every $s \in \mathbb{N} \setminus \{1\}$ and every Archimedean vector lattice over \mathbb{K} . We also generalize Theorem 3.1 of [7] to $s_{\mathbb{K}}$ -linear maps of order bounded variation.

Central to the theory of s -powers are orthosymmetric s -morphisms.

Definition 5.1. *For Archimedean vector lattices E_1, \dots, E_s, F over \mathbb{K} , a map $T : \times_s E \rightarrow F$ is called orthosymmetric if $T(f_1, \dots, f_s) = 0$ whenever there exist $i, j \in \{1, \dots, s\}$ such that $|f_i| \wedge |f_j| = 0$.*

Definition 5.2. *Let E be an Archimedean vector lattice over \mathbb{K} and let $s \in \mathbb{N} \setminus \{1\}$. We call a pair $(E^{\bar{\otimes}}, \bar{\otimes})$ an s -power of E if the following hold.*

- (1) $E^{\bar{\otimes}}$ is an Archimedean vector lattice over \mathbb{K} .
- (2) $\bar{\otimes} : \times_s E \rightarrow E^{\bar{\otimes}}$ is an orthosymmetric vector lattice s -morphism.
- (3) For every Archimedean vector lattice F over \mathbb{K} and every orthosymmetric s -morphism $T : \times_s E \rightarrow F$, there exists a unique vector lattice homomorphism $T^{\bar{\otimes}}$ such that $T^{\bar{\otimes}} \circ \bar{\otimes} = T$.

We address the existence and uniqueness of s -powers for Archimedean vector lattices over \mathbb{K} in our next theorem, which extends Theorem 3.2 in [5]. We denote $E \bar{\otimes} \dots \bar{\otimes} E$ (s times) by $\bar{\otimes}_s E$.

Theorem 5.3. *If E is an Archimedean vector lattice over \mathbb{K} and $s \in \mathbb{N} \setminus \{1\}$ then there exists an s -power of E , unique up to vector lattice isomorphism.*

Proof. The proof for $\mathbb{K} = \mathbb{R}$ is the content of the proof of Theorem 3.2 in [5], so we assume $\mathbb{K} = \mathbb{C}$. Let E be an Archimedean vector lattice over \mathbb{C} and let I be the smallest uniformly closed ideal of $\bar{\otimes}_s E$ that contains

$$\{f_1 \bar{\otimes} \dots \bar{\otimes} f_s : f_1, \dots, f_s \in E \text{ and } |f_j| \wedge |f_k| = 0 \text{ for some } j, k \in \{1, \dots, s\}\}.$$

Given $f \in \bar{\otimes}_s E$, we denote the equivalence class of f in $(\bar{\otimes}_s E)/I$ by $[f]$. Then $(\bar{\otimes}_s E)/I$ is a vector space over \mathbb{C} under the operations $[f] + [g] = [f + g]$ and $\lambda[f] = [\lambda f]$ for all $\lambda \in \mathbb{C}$. From $[f + ig] = [f] + i[g]$, we see that $(\bar{\otimes}_s E)/I = ((\bar{\otimes}_s E)_\rho / I_\rho)_\mathbb{C}$ (see page 198 of [26]). Also, I_ρ is a uniformly closed ideal in $(\bar{\otimes}_s E)_\rho$, and thus $(\bar{\otimes}_s E)_\rho / I_\rho$ is an Archimedean vector lattice over \mathbb{R} . Let $p : (\bar{\otimes}_s E)_\rho \rightarrow (\bar{\otimes}_s E)_\rho / I_\rho$ be the natural vector lattice homomorphism, i.e., $p(f) = [f]$ for all $f \in (\bar{\otimes}_s E)_\rho$. Let $[f], [g] \in (\bar{\otimes}_s E)_\rho / I_\rho$. By Theorem 3.12 in [6], $\mu_{2,4}([f], [g]) = [\mu_{2,4}(f, g)]$, which is in $(\bar{\otimes}_s E)_\rho / I_\rho$ since $(\bar{\otimes}_s E)_\rho$ is square mean complete. Hence $(\bar{\otimes}_s E)_\rho / I_\rho$ is also square mean complete and $(\bar{\otimes}_s E)/I$ is an Archimedean complex vector lattice. Next, let $q : \bar{\otimes}_s E \rightarrow (\bar{\otimes}_s E)/I$ be the natural vector lattice homomorphism from $\bar{\otimes}_s E$ to $(\bar{\otimes}_s E)/I$. Following Theorem 4 in [8], it is straightforward to show that $((\bar{\otimes}_s E)/I, q \circ \bar{\otimes})$ is an s -power of E . The proof of the uniqueness of s -powers is the same as the real case. \square

The Archimedean complex vector lattice tensor product can also be used to obtain results for multilinear maps over \mathbb{K} of ordered bounded variation. We start with some definitions.

Let E be an Archimedean vector lattice over \mathbb{K} and let $a \in E^+$. A *partition* of a is a finite sequence $\{x_k\}_{k=1}^n$ in E^+ such that $\sum_{k=1}^n x_k = a$. As in [7], we denote the set of all partitions of a by $\prod a$ and abbreviate a partition $\{x_k\}_{k=1}^n$ of a by x , which explains the shorthand $x \in \prod a$.

Definition 5.4. Let E_1, \dots, E_s, F be Archimedean vector lattices over \mathbb{K} . We say that an $s_\mathbb{K}$ -linear map $T : \times_{k=1}^s E_k \rightarrow F$ is of order bounded variation if for all $a_k \in E_k^+$ ($k \in \{1, \dots, s\}$) the set

$$\left\{ \sum_{n_1, \dots, n_s} |T(x_{n_1}^1, \dots, x_{n_s}^s)| : x^k \in \prod a_k (k = 1, \dots, s) \right\}$$

is order bounded. We denote by $\mathcal{L}_{bv}(E_1, \dots, E_s; F)$ the space of all $s_\mathbb{K}$ -linear maps of order bounded variation from $\times_{k=1}^s E_k$ into F .

Definition 5.5. Let V be a vector space over \mathbb{K} . We call $K \subseteq V$ a cone in V if $K + K \subseteq K$, $\lambda K \subseteq K$ for every $\lambda \in \mathbb{K}^+$, and $K \cap (-K) = \{0\}$. The pair (V, K) where V is a vector space over \mathbb{K} and K is a cone in V is called an ordered vector space over \mathbb{K} .

For example, if E_1, \dots, E_s, F are Archimedean vector lattices over \mathbb{K} then $\mathcal{L}_{bv}(E_1, \dots, E_s; F)$ is a vector space over \mathbb{K} and the set of all positive maps in $\mathcal{L}_{bv}(E_1, \dots, E_s; F)$, which we denote by $\mathcal{L}_{bv}^+(E_1, \dots, E_s; F)$, is a cone.

Let $(V_1, K_1), \dots, (V_s, K_s), (W, K)$ be ordered vector spaces over \mathbb{K} . We say that a map $T : \times_{k=1}^s V_k \rightarrow W$ is *positive* if $T(\times_{k=1}^s K_k) \subseteq K$. If a positive \mathbb{K} -linear map T is bijective and has a positive inverse, we call T an *ordered vector space isomorphism*. If there exists an ordered vector space isomorphism between ordered vector spaces (V, K) and (W, K') over \mathbb{K} we say that (V, K) and (W, K') are *isomorphic as ordered vector spaces*.

For the proof of the following lemma, let E be an Archimedean vector lattice over \mathbb{K} , let (V, K) be an ordered vector space over \mathbb{K} , and let $\phi : E \rightarrow V$ be an ordered vector space isomorphism with respect to the cones E^+ and K . It is readily checked that the map $m : V \rightarrow V$ defined by $m(v) = \phi(|\phi^{-1}(v)|)$ is an Archimedean modulus on V with $m(V) = K$.

Lemma 5.6. *If E is an Archimedean vector lattices over \mathbb{K} , (V, K) is an ordered vector space over \mathbb{K} , and $\phi : E \rightarrow V$ is an ordered vector space isomorphism with respect to the cones E^+ and K then*

- (1) *V is an Archimedean vector lattice over \mathbb{K} with K as positive cone, and*
- (2) *ϕ is a vector lattice isomorphism.*

The following result generalizes Proposition 3.2(4) in [24] as well as Theorem 3.1 in [7].

Theorem 5.7. *Let E_1, \dots, E_s, F be Archimedean vector lattices over \mathbb{K} with F Dedekind complete.*

- (1) *For any $s_{\mathbb{K}}$ -linear map of order bounded variation $T : \times_{k=1}^s E_k \rightarrow F$ there exists a unique order bounded \mathbb{K} -linear map $T^{\otimes} : \bar{\otimes}_{k=1}^s E_k \rightarrow F$ such that $T(f_1, \dots, f_s) = T^{\otimes}(f_1 \bar{\otimes} \dots \bar{\otimes} f_s)$ for every $f_k \in E_k$ ($k \in \{1, \dots, s\}$).*
- (2) *$\mathcal{L}_{bv}(E_1, \dots, E_s; F)$ is a Dedekind complete Archimedean vector lattice over \mathbb{K} and the correspondence $T \mapsto T^{\otimes}$ is a vector lattice isomorphism from $\mathcal{L}_{bv}(E_1, \dots, E_s; F)$ onto $\mathcal{L}_b(\bar{\otimes}_{k=1}^s E_k, F)$.*
- (3) *For $T \in \mathcal{L}_{bv}(E_1, \dots, E_s; F)$,*

$$|T|(a_1, \dots, a_s) = \sup \left\{ \sum_{n_1, \dots, n_s} |T(x_{n_1}^1, \dots, x_{n_s}^s)| : x^k \in \prod a_k \ (k \in \{1, \dots, s\}) \right\}$$
for all $a_k \in E_k^+$ ($k \in \{1, \dots, s\}$).

Proof. For $\mathbb{K} = \mathbb{R}$ the result is Theorem 3.1 in [7]. We thus assume $\mathbb{K} = \mathbb{C}$.

- (1) For the uniqueness, suppose that $T : \times_{k=1}^s E_k \rightarrow F$ is an $s_{\mathbb{C}}$ -linear map of order bounded variation, and assume that S_1, S_2 are complex order bounded linear maps from

$\bar{\otimes}_{k=1}^s E_k$ to F such that $T = S_1 \circ \bar{\otimes}$ and $T = S_2 \circ \bar{\otimes}$. Then $S_1 - S_2 = 0$ identically on $\bar{\otimes}_{k=1}^s E_k$. By relatively uniform density, $S_1 - S_2 = 0$ on $(\bar{\otimes}_{k=1}^s E_k)_2$, where $(\bar{\otimes}_{k=1}^s E_k)_2$ denotes the pseudo uniform closure of $\bar{\otimes}_{k=1}^s E_k$ in $\bar{\otimes}_{k=1}^s E_k$ (see Section 2). By relatively uniform density again, $S_1 - S_2 = 0$ on $(\bar{\otimes}_{k=1}^s E_k)_3$, where $(\bar{\otimes}_{k=1}^s E_k)_3$ denotes the pseudo uniform closure of $(\bar{\otimes}_{k=1}^s E_k)_2$ in $\bar{\otimes}_{k=1}^s E_k$. From Theorem 4.7(3), we have $(\bar{\otimes}_{k=1}^s E_k)_3 = \bar{\otimes}_{k=1}^s E_k$. We next turn to the existence. Define $\bar{T}_+(a_1, \dots, a_s) := \sup\{\sum_{n_1, \dots, n_s} |T(x_{n_1}^1, \dots, x_{n_s}^s)| : x^k \in \prod a_k (k \in \{1, \dots, s\})\}$ for every $a_k \in E_k^+$ ($k \in \{1, \dots, s\}$). Like in the proof of Theorem 3.1 of [7], one infers that \bar{T}_+ is additive and positively homogeneous in each variable separately. Therefore, by routine reasoning, T_+ uniquely extends to a positive $s_{\mathbb{R}}$ -linear map $\bar{T} : \times_{k=1}^s E_{k\rho} \rightarrow F_{\rho}$, and subsequently to a positive $s_{\mathbb{C}}$ -linear map $\bar{T}_{\mathbb{C}} : \times_{k=1}^s E_k \rightarrow F$. Then $\bar{T}_{\mathbb{C}} - T$ is also a positive $s_{\mathbb{C}}$ -linear map. By Theorem 4.13 there exists unique positive linear maps $\bar{T}_{\mathbb{C}}^{\bar{\otimes}}$ and $(\bar{T}_{\mathbb{C}} - T)^{\bar{\otimes}}$ from $\bar{\otimes}_{k=1}^s E_k$ into F with $\bar{T}_{\mathbb{C}} = \bar{T}_{\mathbb{C}}^{\bar{\otimes}} \circ \bar{\otimes}$ and $(\bar{T}_{\mathbb{C}} - T) = (\bar{T}_{\mathbb{C}} - T)^{\bar{\otimes}} \circ \bar{\otimes}$. Then for $T^{\bar{\otimes}} := \bar{T}_{\mathbb{C}}^{\bar{\otimes}} - (\bar{T}_{\mathbb{C}} - T)^{\bar{\otimes}}$ we have $T^{\bar{\otimes}} \circ \bar{\otimes} = T$.

(2) The map $\Phi : \mathcal{L}_{bv}(E_1, \dots, E_s; F) \rightarrow \mathcal{L}_b(\bar{\otimes}_{k=1}^s E_k, F)$ is a \mathbb{C} -linear map. Let the map $T : \times_{k=1}^s E_k \rightarrow F$ be a positive s -linear map and let $\sum_{j=1}^n f_j^1 \otimes \dots \otimes f_j^s \in \bar{\otimes}_{k=1}^s E_k$ with $f_j^k \in E_k^+$ for every $j \in \{1, \dots, n\}$ ($k \in \{1, \dots, s\}$). Then $T^{\bar{\otimes}}(\sum_{j=1}^n f_j^1 \otimes \dots \otimes f_j^s) = \sum_{j=1}^n T(f_j^1, \dots, f_j^s) \in F^+$. By relatively uniform density, $T^{\bar{\otimes}}$ is positive on $\bar{\otimes}_{k=1}^s E_k$. Therefore, Φ is positive. By Theorem 4.13 we have that for every positive linear map $S : \bar{\otimes}_{k=1}^s E_k \rightarrow F$, there exists a unique positive map $T \in \mathcal{L}_{bv}(E_1, \dots, E_s; F)$ such that $T^{\bar{\otimes}} = S$. Therefore Φ and Φ^{-1} are ordered vector space isomorphisms with respect to the cones $\mathcal{L}_{bv}^+(E_1, \dots, E_s; F)$ and $(\mathcal{L}_b(\bar{\otimes}_{k=1}^s E_k, F))^+$. It follows from Lemma 5.6 that $\mathcal{L}_{bv}(E_1, \dots, E_s; F)$ is an Archimedean vector lattice over \mathbb{C} with $\mathcal{L}_{bv}^+(E_1, \dots, E_s; F)$ as positive cone. Also by Lemma 5.6, Φ is a vector lattice isomorphism, and so $\mathcal{L}_{bv}(E_1, \dots, E_s; F)$ is Dedekind complete.

(3) Let $a_k \in E_k^+$ ($k \in \{1, \dots, s\}$) and put $\theta \in [0, 2\pi]$. Also let $T \in \mathcal{L}_{bv}(E_1, \dots, E_s; F)$, and let \bar{T} be as in part (1) of this proof. Evidently, we have $\bar{T}(a_1, \dots, a_s) \geq (Re(e^{-i\theta}T))(a_1, \dots, a_s)$ and thus $\bar{T} \geq \sup\{Re(e^{-i\theta}T) : \theta \in [0, 2\pi]\} = |T|$ (see page 188 in [26] for this alternative definition of the modulus). On the other hand, if $x^k \in \prod a_k$ ($k \in \{1, \dots, s\}$) then $\sum_{n_1, \dots, n_s} |T(x_{n_1}^1, \dots, x_{n_s}^s)| \leq \sum_{n_1, \dots, n_s} |T|(x_{n_1}^1, \dots, x_{n_s}^s) = |T|(a_1, \dots, a_s)$. Therefore $\bar{T} \leq |T|$. \square

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